15.401 Finance Theory

MIT Sloan MBA Program

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Lectures 2–3: Present Value Relations

Critical Concepts

- Cashflows and Assets
- The Present Value Operator
- The Time Value of Money
- Special Cashflows: The Perpetuity
- Special Cashflows: The Annuity
- Compounding
- Inflation
- Extensions and Qualifications

Readings:

- Brealey, Myers, and Allen Chapters 2–3

Cashflows and Assets

Key Question: What Is An “Asset”?

- Business entity
- Property, plant, and equipment
- Patents, R&D
- Stocks, bonds, options, …
- Knowledge, reputation, opportunities, etc.

From A Business Perspective, An Asset Is A <u>Sequence</u> of Cashflows

$$\text{Asset}_t \equiv \{\text{CF}_t, \text{CF}_{t+1}, \text{CF}_{t+2}, \ldots\}$$

Examples of Assets as Cashflows

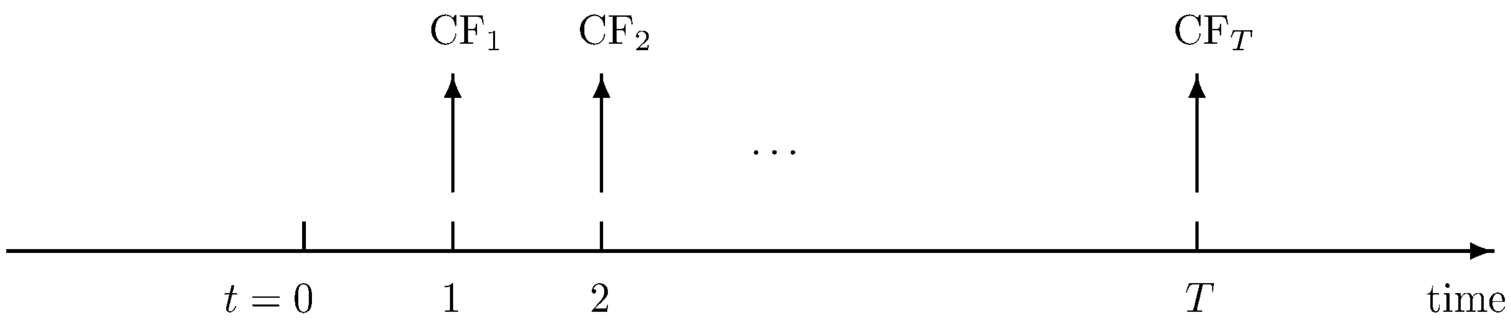
- Boeing is evaluating whether to proceed with development of a new regional jet. You expect development to take 3 years, cost roughly $850 million, and you hope to get unit costs down to $33 million. You forecast that Boeing can sell 30 planes every year at an average price of $41 million.
- Firms in the S&P 500 are expected to earn, collectively, $66 this year and to pay dividends of $24 per share, adjusted to index. Dividends and earnings have grown 6.6% annually (or about 3.2% in real terms) since 1926.
- You were just hired by HP. Your initial pay package includes a grant of 50,000 stock options with a strike price of $24.92 and an expiration date of 10 years. HP's stock price has varied between $16.08 and $26.03 during the past two years.

Cashflows and Assets

Valuing An Asset Requires Valuing A Sequence of Cashflows

- Sequences of cashflows are the "basic building blocks" of finance

$$\text{Value of Asset}_t \quad \equiv \quad V_t(\text{CF}_t, \text{CF}_{t+1}, \text{CF}_{t+2}, \ldots)$$



Always Draw A Timeline To Visualize The Timing of Cashflows

The Present Value Operator

What is V_t?

- What factors are involved in determining the value of any object?
  - Subjective?
  - Objective?
- How is value determined?

There Are Two Distinct Cases

- No Uncertainty
  - We have a complete solution
- Uncertainty
  - We have a partial solution (approximation)
  - The reason: synergies and other interaction effects
- Value is determined the same way, but we want to **understand** how

The Present Value Operator

Key Insight: Cashflows At Different Dates Are Different "Currencies"

- Consider manipulating foreign currencies

$$¥150 + £300 \overset{?}{=} ??$$

The Present Value Operator

Key Insight: Cashflows At Different Dates Are Different "Currencies"

- Consider manipulating foreign currencies

$$¥150 + £300 \stackrel{?}{=} ??450$$

- Cannot add currencies without first converting into common currency

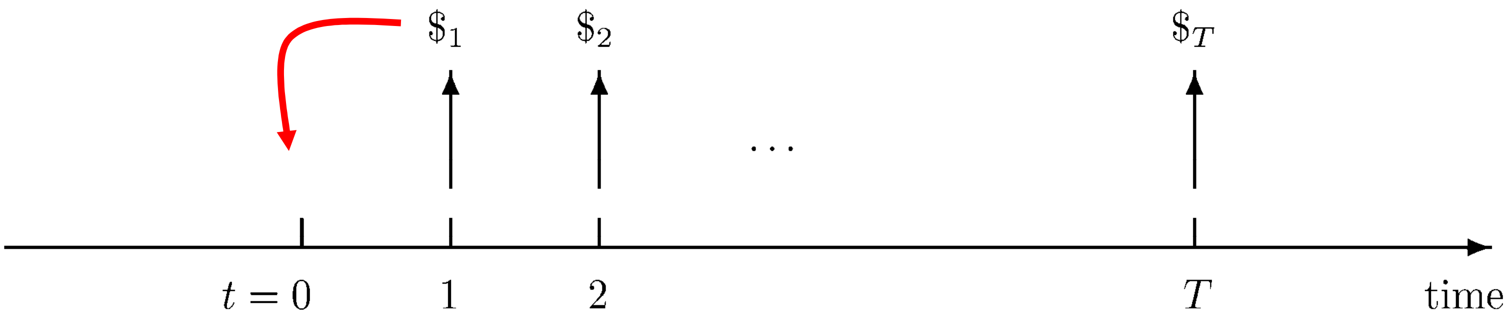
$$¥150 + (£300) \times (153\ ¥ / £) = ¥46,050.00$$
$$(¥150) \times (0.0065\ £ / ¥) + £300 = £\ 300.98$$

- Given exchange rates, either currency can be used as "numeraire"
- Same idea for cashflows of different dates

The Present Value Operator

Key Insight: Cashflows At Different Dates Are Different "Currencies"

- Past and future cannot be combined without first converting them
- Once "exchange rates" are given, combining cashflows is trivial



- A **numeraire** date should be picked, typically t=0 or "today"
- Cashflows can then be converted to **present value**

$$V_0(\mathrm{CF}_1, \mathrm{CF}_2, \mathrm{CF}_3, \ldots) = \left(\frac{\$_1}{\$_0}\right) \times \mathrm{CF}_1 + \left(\frac{\$_2}{\$_0}\right) \times \mathrm{CF}_2 + \cdots$$

The Present Value Operator

Net Present Value: "Net" of Initial Cost or Investment

- Can be captured by date-0 cashflow CF_0

$$V_0(\text{CF}_0, \text{CF}_1, \ldots) \;=\; \text{CF}_0 \;+\; \left(\frac{\$_1}{\$_0}\right) \times \text{CF}_1 \;+\; \left(\frac{\$_2}{\$_0}\right) \times \text{CF}_2 \;+\; \cdots$$

- If there is an initial investment, then $CF_0 < 0$
- Note that any CF_t can be negative (future costs)
- V_0 is a completely general expression for net present value

How Can We Decompose V_0 Into Present Value of Revenues and Costs?

The Present Value Operator

Example:

- Suppose we have the following "exchange rates":

$$\left(\frac{\$_1}{\$_0}\right) = 0.90 \quad , \quad \left(\frac{\$_2}{\$_0}\right) = 0.80$$

- What is the net present value of a project requiring a current investment of \$10MM with cashflows of \$5MM in Year 1 and \$7MM in Year 2?

$$\text{NPV}_0 = -\$10 + \$5 \times 0.90 + \$7 \times 0.80 = \$0.10$$

- Suppose a buyer wishes to purchase this project but pay for it two years from now. How much should you ask for?

The Present Value Operator

Example:

- Suppose we have the following "exchange rates":

$$\left(\frac{\$_1}{\$_0}\right) = 0.90 \quad , \quad \left(\frac{\$_2}{\$_0}\right) = 0.80$$

- What is the net present value of a project requiring an investment of \$8MM in Year 2, with a cashflow of \$2MM immediately and a cashflow of \$5 in Year 1?

$$\text{NPV}_0 = \$2 + \$5 \times 0.90\;0 - \$8 \times 0.80 = \$0.10$$

- Suppose a buyer wishes to purchase this project but pay for it two years from now. How much should you ask for?

The Time Value of Money

Implicit Assumptions/Requirements For NPV Calculations

- Cashflows are known (magnitudes, signs, timing)
- Exchange rates are known
- No frictions in currency conversions

Do These Assumptions Hold in Practice?

- Which assumptions are most often violated?
- Which assumptions are most plausible?

Until Lecture 12, We Will Take These Assumptions As Truth

- Focus now on exchange rates
- Where do they come from, how are they determined?

The Time Value of Money

What Determines The Growth of $1 Over T Years?

- $1 today should be worth more than $1 in the future (why?)
- Supply and demand
- **Opportunity cost of capital r**

$$
\begin{aligned}
\$1 \text{ in Year } 0 &= \$1 \times (1+r) \quad \text{in Year } 1 \\
\$1 \text{ in Year } 0 &= \$1 \times (1+r)^2 \text{ in Year } 2 \\
&\vdots \\
\$1 \text{ in Year } 0 &= \$1 \times (1+r)^T \text{ in Year } T
\end{aligned}
$$

- Equivalence of $1 today and any other single choice above
- Other choices are **future values** of $1 today

The Time Value of Money

What Determines The Value Today of $1 In Year-T?

- $1 in Year-T should be worth less than $1 today (why?)
- Supply and demand
- Opportunity cost of capital r

$$
\begin{aligned}
\$1/(1+r) \ \text{ in Year 0} &= \text{\$1 in Year 1} \\
\$1/(1+r)^2 \ \text{ in Year 0} &= \text{\$1 in Year 2} \\
&\vdots \\
\$1/(1+r)^T \ \text{ in Year 0} &= \text{\$1 in Year } T
\end{aligned}
$$

- These are our "exchange rates" ($\$_t/\$_0$) or **discount factors**

We Now Have An Explicit Expression for V_0:

$$V_0 = \mathsf{CF}_0 + \frac{1}{(1+r)} \times \mathsf{CF}_1 + \frac{1}{(1+r)^2} \times \mathsf{CF}_2 + \cdots$$

$$V_0 = \mathsf{CF}_0 + \frac{\mathsf{CF}_1}{(1+r)} + \frac{\mathsf{CF}_2}{(1+r)^2} + \cdots$$

- Using this expression, any cashflow can be valued!
- **Take positive-NPV projects, reject negative NPV-projects**
- Projects ranked by magnitudes of NPV
- All capital budgeting and corporate finance reduces to this expression
- However, we still require many assumptions (perfect markets)

The Time Value of Money

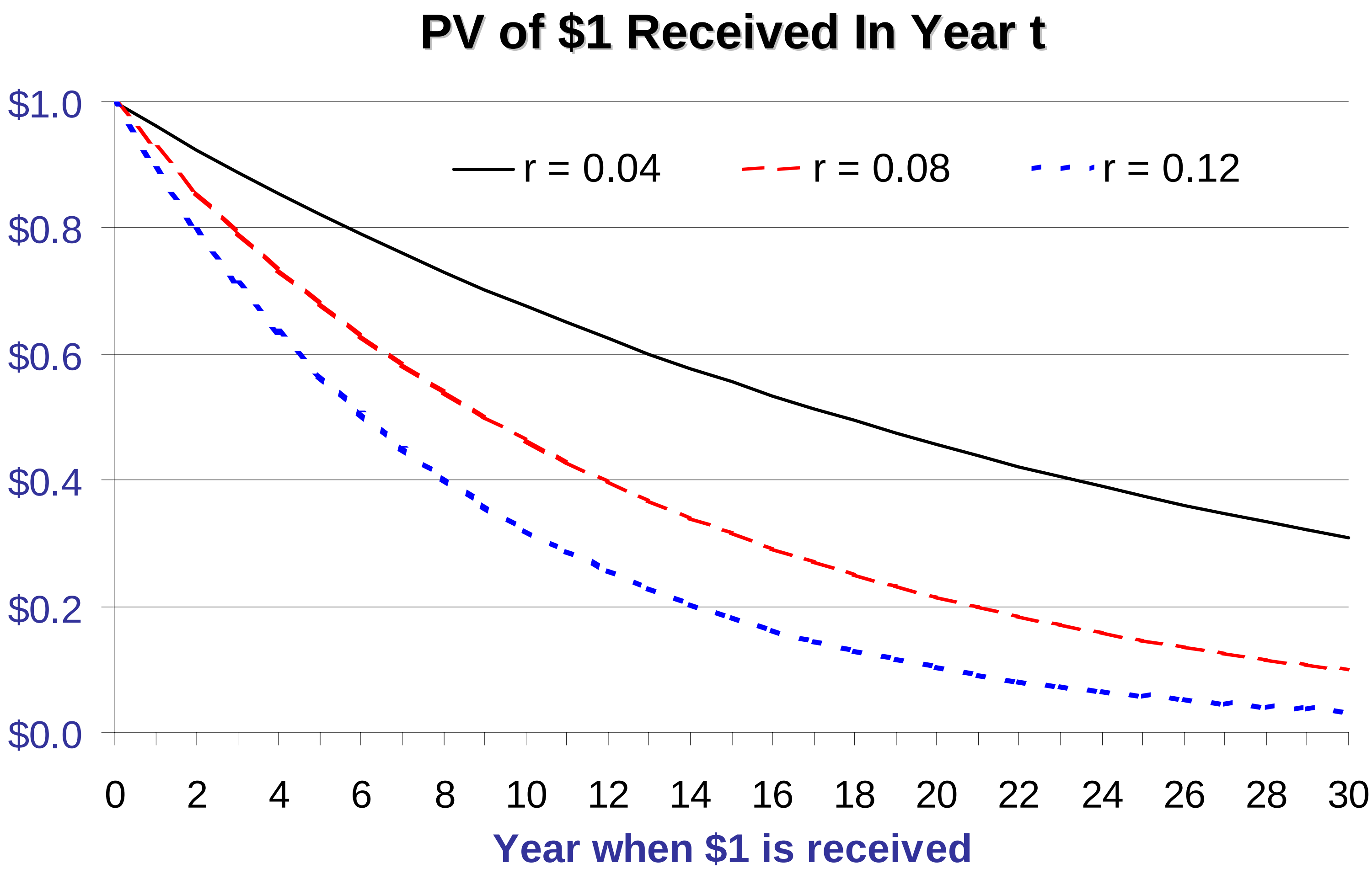
Example:

- Suppose you have \$1 today and the interest rate is 5%. How much will you have in …

1 year …	$\$1 \times 1.05 = \$1.05$
2 years …	$\$1 \times 1.05 \times 1.05 = \$1.103$
3 years …	$\$1 \times 1.05 \times 1.05 \times 1.05 = \$1.158$

- \$1 today is equivalent to $\$1 \times (1+r)^t$ in t years
- \$1 in t years is equivalent to $\$\frac{1}{(1+r)^t}$ today

The Time Value of Money



The Time Value of Money

Example:

Your firm spends $800,000 annually for electricity at its Boston headquarters. Johnson Controls offers to install a new computer-controlled lighting system that will reduce electric bills by $90,000 in each of the next three years. If the system costs $230,000 fully installed, is this a good investment?

Lighting System*

Year	0	1	2	3
Cashflow	-230,000	90,000	90,000	90,000

* Assume the cost savings are known with certainty and the interest rate is 4%

The Time Value of Money

Example:

Lighting System

Year	0	1	2	3
Cashflow	-230,000	90,000	90,000	90,000
÷		1.04	$(1.04)^2$	$(1.04)^3$
PV	-230,000	86,538	83,210	80,010

NPV = -230,000 + 86,538 + 83,210 + 80,010 = $19,758

- **Go ahead – project looks good!**

The Time Value of Money

Example:

CNOOC recently made an offer of $67 per share for Unocal. As part of the takeover, CNOOC will receive $7 billion in 'cheap' loans from its parent company: a zero-interest, 2-year loan of $2.5 billion and a 3.5%, 30-year loan of $4.5 billion. If CNOOC normal borrowing rate is 8%, how much is the interest subsidy worth?

- Interest Savings, Loan 1: 2.5 × (0.08 – 0.000) = $0.2 billion
- Interest Savings, Loan 2: 4.5 × (0.08 – 0.035) = $0.2 billion

$$\begin{aligned} PV &= \frac{0.4}{(1.08)} + \frac{0.4}{(1.08)^2} + \frac{0.2}{(1.08)^3} + \frac{0.2}{(1.08)^4} + \cdots + \frac{0.2}{(1.08)^{30}} \\ &= \$2.62 \text{ billion} \end{aligned}$$

Special Cashflows: The Perpetuity

Perpetuity Pays Constant Cashflow *C* Forever

- How much is an infinite cashflow of *C* each year worth?
- How can we value it?

$$\text{PV} = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

$$(1+r) \times \text{PV} = C + \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \cdots$$

$$r \times \text{PV} = C$$

$$\text{PV} = \frac{C}{r}$$

Special Cashflows: The Perpetuity

Growing Perpetuity Pays Growing Cashflow *C*(1+*g*)^t Forever

- How much is an infinite growing cashflow of *C* each year worth?
- How can we value it?

$$\text{PV} = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots$$

$$\frac{(1+r)}{(1+g)} \times \text{PV} = \frac{C}{(1+g)} + \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \cdots$$

$$\left[\frac{(1+r)}{(1+g)} - 1\right] \times \text{PV} = \frac{C}{(1+g)}$$

$$\text{PV} = \frac{C}{r-g} \quad , \quad r > g$$

Special Cashflows: The Annuity

Annuity Pays Constant Cashflow *C* For T Periods

- Simple application of V_0

$$\text{PV} = \frac{C}{(1+r)} + \cdots + \frac{C}{(1+r)^T}$$

$$(1+r) \times \text{PV} = C + \frac{C}{(1+r)} + \frac{C}{(1+r)^{T-1}}$$

$$r \times \text{PV} = C - \frac{C}{(1+r)^T}$$

$$\text{PV} = \frac{C}{r} - \frac{C}{r}\frac{1}{(1+r)^T}$$

Special Cashflows: The Annuity

Annuity Pays Constant Cashflow *C* For T Periods

- Sometimes written as a product:

$$\text{PV} \;=\; \frac{C}{r} - \frac{C}{r}\frac{1}{(1+r)^T} \;=\; C \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^T}\right]$$

$$= \; C \times \text{ADF}(r,T)$$

$$\text{ADF}(r,T) \;\equiv\; \frac{1}{r}\left[1 - \frac{1}{(1+r)^T}\right]$$

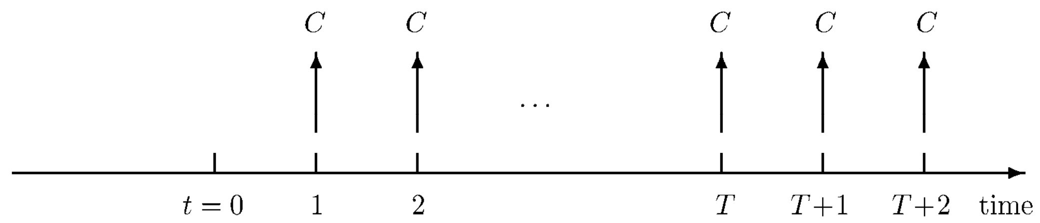
r	T: 1	2	3	4	5	10	15	20	25	30
5%	0.952	1.859	2.723	3.546	4.329	7.722	10.380	12.462	14.094	15.372
6%	0.943	1.833	2.673	3.465	4.212	7.360	9.712	11.470	12.783	13.765
7%	0.935	1.808	2.624	3.387	4.100	7.024	9.108	10.594	11.654	12.409
8%	0.926	1.783	2.577	3.312	3.993	6.710	8.559	9.818	10.675	11.258
9%	0.917	1.759	2.531	3.240	3.890	6.418	8.061	9.129	9.823	10.274
10%	0.909	1.736	2.487	3.170	3.791	6.145	7.606	8.514	9.077	9.427
11%	0.901	1.713	2.444	3.102	3.696	5.889	7.191	7.963	8.422	8.694
12%	0.893	1.690	2.402	3.037	3.605	5.650	6.811	7.469	7.843	8.055
13%	0.885	1.668	2.361	2.974	3.517	5.426	6.462	7.025	7.330	7.496
14%	0.877	1.647	2.322	2.914	3.433	5.216	6.142	6.623	6.873	7.003
15%	0.870	1.626	2.283	2.855	3.352	5.019	5.847	6.259	6.464	6.566
16%	0.862	1.605	2.246	2.798	3.274	4.833	5.575	5.929	6.097	6.177
17%	0.855	1.585	2.210	2.743	3.199	4.659	5.324	5.628	5.766	5.829
18%	0.847	1.566	2.174	2.690	3.127	4.494	5.092	5.353	5.467	5.517
19%	0.840	1.547	2.140	2.639	3.058	4.339	4.876	5.101	5.195	5.235
20%	0.833	1.528	2.106	2.589	2.991	4.192	4.675	4.870	4.948	4.979

Special Cashflows: The Annuity

Annuity Pays Constant Cashflow *C* For T Periods

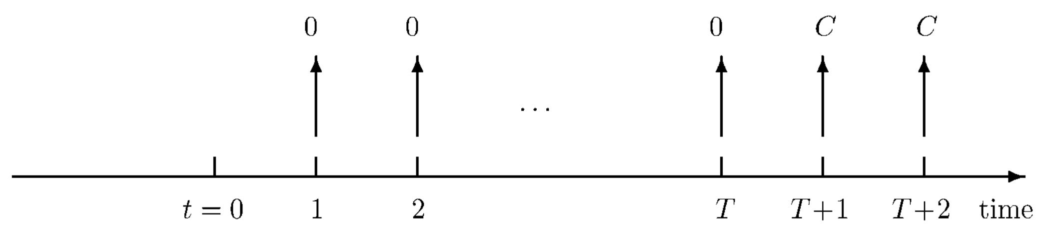
- Related to perpetuity formula

Perpetuity



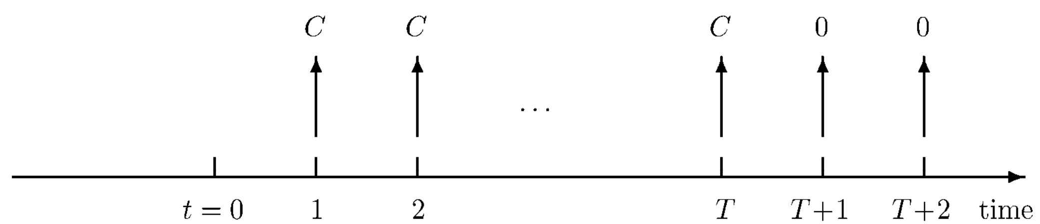
Minus

Date-T Perpetuity



Equals

T-Period Annuity



Special Cashflows: The Annuity

Example:

You just won the lottery and it pays $100,000 a year for 20 years. Are you a millionaire? Suppose that r = 10%.

$$\begin{aligned} PV &= 100,000 \times \frac{1}{0.10}\left(1 - \frac{1}{1.10^{20}}\right) \\ &= 100,000 \times 8.514 = 851,356 \end{aligned}$$

- What if the payments last for 50 years?

$$\begin{aligned} PV &= 100,000 \times \frac{1}{0.10}\left(1 - \frac{1}{1.10^{50}}\right) \\ &= 100,000 \times 9.915 = 991,481 \end{aligned}$$

- How about forever (a perpetuity)?

$$PV = 100,000/0.10 = 1,000,000$$

Compounding

Interest May Be Credited/Charged More Often Than Annually

- Bank accounts: daily
- Mortgages and leases: monthly
- Bonds: semiannually
- **Effective annual rate** may differ from **annual percentage rate**
- Why?

Typical Compounding Conventions:

- Let r denote APR, n periods of compounding
- ***r/n* is per-period rate for each period**
- Effective annual rate (EAR) is

$$r_{\text{EAR}} \equiv (1 + r/n)^n - 1$$

10% Compounded Annually, Semi-Annually, Quarterly, and Monthly

Month	$1,000	$1,000	$1,000	$1,000
1				$1,008
2				$1,017
3			$1,025	$1,025
4				$1,034
5				$1,042
6		$1,050	$1,051	$1,051
7				$1,060
8				$1,069
9			$1,077	$1,078
10				$1,087
11				$1,096
12	$1,100	$1,103	$1,104	$1,105

Compounding

Example:

Car loan—'Finance charge on the unpaid balance, *computed daily*, at the rate of 6.75% per year.'

If you borrow $10,000, how much would you owe in a year?

Daily interest rate = 6.75 / 365 = 0.0185%

Day 1: Balance = 10,000.00 × 1.000185 = 10,001.85

Day 2: Balance = 10,001.85 × 1.000185 = 10,003.70

… …

Day 365: Balance = 10,696.26 × 1.000185 = 10,698.24

EAR = 6.982% > 6.750%

Inflation

What Is Inflation?

- Change in real purchasing power of $1 over time
- Different from time-value of money (how?)
- For some countries, inflation is extremely problematic
- How to quantify its effects?

$$\begin{aligned}\text{Wealth } W_t &\Leftrightarrow \text{Price Index } I_t \\ \text{Wealth } W_{t+k} &\Leftrightarrow \text{Price Index } I_{t+k} \\ \text{Increase in Cost of Living} &\equiv I_{t+k}/I_t \quad = \quad (1+\pi)^k \\ \text{“Real Wealth” } \widetilde{W}_{t+k} &\equiv W_{t+k}/(1+\pi)^k\end{aligned}$$

Inflation

$$\text{"Real Wealth"} \quad \widetilde{W}_{t+k} \equiv W_{t+k}/(1+\pi)^k$$

$$\text{"Real Return"} \quad (1+r_{\text{real}})^k \equiv \frac{\widetilde{W}_{t+k}}{W_t}$$

$$= \frac{W_{t+k}}{W_t}\frac{1}{(1+\pi)^k} = \frac{(1+r_{\text{nominal}})^k}{(1+\pi)^k}$$

$$r_{\text{real}} = \frac{1+r_{\text{nominal}}}{1+\pi} - 1$$

$$\approx r_{\text{nominal}} - \pi$$

$$r_{\text{real}} \approx r_{\text{nominal}} - \pi$$

Inflation

For NPV Calculations, Treat Inflation Consistently

- Discount **real cashflows** using **real interest rates**
- Discount **nominal cashflows** using **nominal interest rates**
  - Nominal cashflows $\Rightarrow$ expressed in actual-dollar cashflows
  - Real cashflows $\Rightarrow$ expressed in constant purchasing power
  - Nominal rate $\Rightarrow$ actual prevailing interest rate
  - Real rate $\Rightarrow$ interest rate adjusted for inflation

Inflation

Example:

This year you earned $100,000. You expect your earnings to grow 2% annually, in real terms, for the remaining 20 years of your career. Interest rates are currently 5% and inflation is 2%. What is the present value of your income?

Real Interest Rate = 1.05 / 1.02 – 1 = 2.94%

Real Cashflows

Year	1	2	…	20
Cashflow	102,000	104,040	…	148,595
÷	1.0294	1.02942^2	…	1.02942^{20}
PV	99,086	98,180	…	83,219

Present Value = $1,818,674

Extensions and Qualifications

- Taxes
- Currencies
- Term structure of interest rates
- Forecasting cashflows
- Choosing the right discount rate (risk adjustments)

Key Points

- Assets are sequences of cashflows
- Date-t cashflows are different from date-(t+k) cashflows
- Use “exchange rates” to convert one type of cashflow into another
- PV and FV related by “exchange rates”
- Exchange rates are determined by supply/demand
- Opportunity cost of capital: expected return on equivalent investments in financial markets
- For NPV calculations, visualize cashflows first
- Decision rule: accept positive NPV projects, reject negative ones
- Special cashflows: perpetuities and annuities
- Compounding
- Inflation
- Extensions and Qualifications

Additional References

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- Copeland, T., Weston, F. and K. Shastri, 2003, *Financial Theory and Corporate Policy,* (4th Edition). Reading, MA: Addison-Wesley.

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